The Conformal Anomaly in General Rank 1 Symmetric Spaces and Associated Operator Product

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Abstract:

We compute the one-loop effective action and the conformal anomaly associated with the product $\otimes_p \mathcal{L}_p$ of the Laplace type operators $\mathcal{L}_p$, $p = 1, 2$, acting in irreducible rank 1 symmetric spaces of non-compact type. The explicit form of the zeta functions and the conformal anomaly of the stress-energy momentum tensor is derived.

1 Introduction

Recently the important role of the multiplicative anomaly has been recognized in physics [1,2,3,4,5,6,7,8,9,10,11,12,13,14]. The anomaly associated with multiplicative properties of regularized determinants of (pseudo-) differential operators can be expressed by means of the non-commutative residue, the Wodzicki residue [15] (see also Refs. [16,17]). The Wodzicki residue, which is the unique extension of the Dixmier trace to the wider class of (pseudo-) differential operators [1,4], has been considered within the non-commutative geometrical approach to the standard model of the electroweak interactions [2,3,9,10,

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11, 12] and the Yang-Mills action functional. Some recent papers along these lines can be found in Refs. [5, 7, 8]. Wodzicki's formulae have been used also for dealing with the singularity structure of the zeta functions [13] and the commutator anomalies of current algebras [6].

Therefore it is natural to investigate the multiplicative properties of differential operators as well as properties of their determinants. The product of two (or more) differential operators of Laplace type can arise in higher derivative field theories (for example, in higher derivative quantum gravity [18, 19]). The partition function corresponding to the product of two elliptic second order differential operators for the simplest $O(2)$ invariant model of self-interacting charged fields in $\mathbf{R}^4$ [20] has been derived recently in Ref. [14]. The global additive and multiplicative properties of Laplace type operators acting in irreducible rank 1 symmetric spaces and the explicit form of the multiplicative anomaly have been derived in Ref. [21].

Under such circumstances we should note that the conformal deformations of a metric and the corresponding conformal anomaly can also play an important role in quantum theories with higher derivatives. It is well known that evaluation of the conformal anomaly is actually possible only for even dimensional spaces and up to now its computation is extremely involved. The general structure of such anomaly in curved d-dimensional spaces (d even) has been studied in Ref. [22]. We briefly mention here analysis related to this phenomenon for constant curvature spaces. The conformal anomaly calculation for d-dimensional sphere can be found in Ref. [23, 24]. The explicit computation of the anomaly (of the stress-energy tensor) for scalar and spinor quantum fields in d-dimensional compact hyperbolic spaces has been carried out in Ref. [25] (see also Refs. [26, 27]) using zeta-function regularization and the Selberg trace formula techniques.

The purpose of this paper is to analyze a contribution to the effective action (in general form) and the conformal anomaly associated with the product $(\otimes_p \mathcal{L}_p)$, where $\mathcal{L}_p$, $p = 1, 2$, are the Laplace type operators acting in general rank 1 symmentric spaces.

The contents of the paper are the following. In Sect. 2 we review the relevant information on the irreducible rank 1 symmetric spaces of non-compact type and the spectral zeta function $\zeta(s|\mathcal{L})$. The explicit form of the zeta function $\zeta(s| \otimes_p \mathcal{L}_p)$ and its derivative (at $s = 0$), the one-loop effective action $W^{(1)}$, and the conformal anomaly of the stress-energy tensor $< T_\mu^\mu(x) >$ related to the operator product are given in Sect. 3. Finally we end with some conclusions in Sect. 4.

2 Irreducible Rank 1 Symmetric Space Forms $\Gamma \backslash G / K$ and the Spectral Zeta Function

We shall be working with irreducible rank 1 symmetric spaces $M \equiv X = G/K$ of non-compact type. Thus G will be a connected non-compact simple split rank 1 Lie group with finite center and $K \subset G$ a maximal compact subgroup. Let $\Gamma \subset G$ be a discrete, co-compact, torsion free subgroup.

Let χ be a finite-dimensional unitary representation of Γ, let $\{\lambda_l\}_{l=0}^\infty$ be the set of eigenvalues of the second-order operator of Laplace type $L = -\Delta_\Gamma$ acting on smooth sections of the vector bundle over $\Gamma \backslash X$ induced by χ, and let $n_l(\chi)$ denote the multiplicity of λ_l.

We shall need further a suitable regularization of the determinant of an elliptic differential operator, and shall make the choice of zeta-function regularization (see Eq. (3.1)). The zeta function associated with the operator $\mathcal{L} \equiv L + b$ has the form

$$\zeta(s|\mathcal{L}) = \sum_l n_l(\chi)\{\lambda_l + b\}^{-s}; \tag{2.1}$$

here b is an arbitrary constant (an endomorphism of the vector bundle over $\Gamma \backslash X$); $\zeta(s|\mathcal{L})$ is a well-defined analytic function for $\operatorname{Re} s > \dim(M)/2$, and can be analytically continued to a meromorphic function on the complex plane $\mathbf{C}$, regular at $s = 0$. One can define the heat kernel of the elliptic operator $\mathcal{L}$ by

$$\omega_{\Gamma_p}(t) \equiv \operatorname{Tr}\left(e^{-t\mathcal{L}_p}\right) = \frac{-1}{2\pi i}\operatorname{Tr}\int_{\mathcal{C}_0} dz e^{-zt}(z - \mathcal{L}_p)^{-1}, \tag{2.2}$$

where $\mathcal{C}_0$ is an arc in the complex plane $\mathbf{C}$. By standard results in operator theory there exist $\epsilon, \delta > 0$ such that for $0 < t < \delta$ the heat kernel expansion holds

$$\omega_\Gamma(t; b, \chi) = \sum_{l=0}^{\infty} n_l(\chi)e^{-(\lambda_l + b)t} = \sum_{0 \le l \le l_0} a_l(\mathcal{L})t^{-l} + O(t^\epsilon). \tag{2.3}$$

The following representations of X up to local isomorphism can be chosen

$$X = \begin{bmatrix} SO_1(n,1)/SO(n) & (I) \\ SU(n,1)/U(n) & (II) \\ SP(n,1)/(SP(n) \otimes SP(1)) & (III) \\ F_{4(-20)}/Spin(9) & (IV) \end{bmatrix}, \tag{2.4}$$

where $n \ge 2$, and $F_{4(-20)}$ is the unique real form of F_4 (with Dynkin diagram $\circ - \circ = \circ - \circ$) for which the character $(\dim X - \dim K)$ assumes the value (-20) [28]. We assume that if G_1 or $G_2 = SO(m,1)$ or $SU(q,1)$ then m is even and q is odd.

The suitable Harish-Chandra-Plancherel measure is given as follows:

$$|C(r)|^{-2} = C_G \pi r P(r) \tanh(a(G)r) = C_G \pi \sum_{l=0}^{\frac{d}{2}-1} a_{2l} r^{2l+1} \tanh(a(G)r), \tag{2.5}$$

where

$$a(G) = \begin{bmatrix} \pi & \text{for } G = SO_1(2n,1) \\ \frac{\pi}{2} & \text{for } G = SU(q,1), \quad q \text{ odd} \\ & \text{or } G = SP(m,1), \quad F_{4(-20)} \end{bmatrix}, \tag{2.6}$$

while C_G is some constant depending on G, and where the $P(r)$ are even polynomials (with suitable coefficients a_{2l}) of degree $d - 2$ for $G \ne SO(2n+1,1)$, and of degree $d - 1 = 2n$ for $G = SO_1(2n+1,1)$ [27, 29].

3 Anomaly Related to the Laplace Type Operator Product

3.1 The One-Loop Effective Action

In this section we are interested in multiplicative properties $\otimes \mathcal{L}_p$ of the second-order operators of Laplace type $\mathcal{L}_p, p = 1, 2$, related with the one-loop effective action in quantum field theory. We shall assume a ζ-regularization determinants, i.e.

$$\det{}_\zeta(\mathcal{L}_p) \stackrel{def}{=} \exp\left(-\frac{\partial}{\partial s}\zeta(s = 0|\mathcal{L}_p)\right). \tag{3.1}$$

To start with, let us recall the general formalism enabling the treatment of the one-loop effective action. Let the data (G, K, Γ) be as in Sect. 2, therefore G being one of the four groups of Eq. (2.4). The trace formula holds [30, 31]

$$\omega_\Gamma(t; b, \chi) = V \int_{\mathbf{R}} dr e^{-(r^2+b+\rho_0^2)t} |C(r)|^{-2} + \theta_\Gamma(t; b, \chi), \tag{3.2}$$

where, by definition,

$$V \stackrel{def}{=} \frac{1}{4\pi}\chi(1)\mathrm{vol}(\Gamma\backslash G), \tag{3.3}$$

where χ is a finite-dimensional unitary representation (or a character) of Γ, and the number ρ_0 is associated with the positive restricted (real) roots of G (with multiplicity) with respect to a nilpotent factor N of G in an Iwasawa decomposition $G = KAN$. One has $\rho_0 = (n-1)/2, n, 2n+1, 11$ in the cases $(I) - (IV)$ respectively in Eq. (2.4). Finally the function $\theta_\Gamma(t; b, \chi)$ is defined as follows

$$\theta_\Gamma(t; b, \chi) \stackrel{def}{=} \frac{1}{\sqrt{4\pi t}} \sum_{\gamma \in C_\Gamma - \{1\}} \chi(\gamma)t_\gamma j(\gamma)^{-1}C(\gamma)e^{-(bt+\rho_0^2 t + t_\gamma^2/(4t))}, \tag{3.4}$$

for a function $C(\gamma)$, $\gamma \in \Gamma$, defined on $\Gamma - \{1\}$ by

$$C(\gamma) \stackrel{def}{=} e^{-\rho_0 t_\gamma} |\det{}_{n_0}\left(\mathrm{Ad}(m_\gamma e^{t_\gamma H_0})^{-1} - 1\right)|^{-1}. \tag{3.5}$$

The notation used in Eqs. (3.4) and (3.5) is the following. Let a_0, n_0 denote the Lie algebras of A, N. Since the rank of G is 1, $\dim a_0 = 1$ by definition, say $a_0 = \mathbf{R}H_0$ for a suitable basis vector H_0. One can normalize the choice of H_0 by $\beta(H_0) = 1$, where $\beta : a_0 \mapsto \mathbf{R}$ is the positive root which defines n_0; for more detail see Ref. [29]. Since Γ is torsion free, each $\gamma \in \Gamma - \{1\}$ can be represented uniquely as some power of a primitive element $\delta : \gamma = \delta^{j(\gamma)}$ where $j(\gamma) \geq 1$ is an integer and δ cannot be written as γ_1^j for $\gamma_1 \in \Gamma$, $j > 1$ an integer. Taking $\gamma \in \Gamma$, $\gamma \neq 1$, one can find $t_\gamma > 0$ and $m_\gamma \in K$ satisfying $m_\gamma a = am_\gamma$ for every $a \in A$ such that γ is G conjugate to $m_\gamma \exp(t_\gamma H_0)$, namely for some $g \in G$, $g\gamma g^{-1} = m_\gamma \exp(t_\gamma H_0)$. For Ad denoting the adjoint representation of G on its complexified Lie algebra, one can compute t_γ as follows [30]

$$e^{t_\gamma} = \max\{|c| \,| c = \text{an eigenvalue of } \mathrm{Ad}(\gamma)\}, \tag{3.6}$$

in case $G = SO_1(m, 1)$, with $|c|$ replaced by $|c|^{1/2}$ in the other cases of Eq. (2.4).

The spectral zeta function associated with the product $\otimes \mathcal{L}_p$ has the form

$$\zeta(s|\bigotimes_p \mathcal{L}_p) = \sum_{j\geq 0} n_j \prod_p^2 (\lambda_j + b_p)^{-s}. \tag{3.7}$$

We shall always assume that $b_1 \neq b_2$, say $b_1 > b_2$. If $b_1 = b_2$ then $\zeta(s|\bigotimes_p \mathcal{L}_p) = \zeta(2s|\mathcal{L})$ is a well-known function. For $b_1, b_2 \in \mathbf{R}$, set $b_+ \stackrel{def}{=} (b_1+b_2)/2$, $b_- \stackrel{def}{=} (b_1-b_2)/2$, thus $b_1 = b_+ + b_-$ and $b_2 = b_+ - b_-$.

The zeta function can be written as follows [21]

$$\zeta(s|\bigotimes_p \mathcal{L}_p) = (2b_-)^{\frac{1}{2}-s}\frac{\sqrt{\pi}}{\Gamma(s)}\int_0^\infty dt$$

$$\times \left[\frac{\chi(1)\mathrm{vol}(\Gamma\backslash G)}{4\pi}\int_{\mathbf{R}} dr e^{-(r^2+b_++\rho_0^2)t}|C(r)|^{-2} + \theta_\Gamma(t)\right]I_{s-\frac{1}{2}}(b_-t)t^{s-\frac{1}{2}}. \qquad (3.8)$$

where $I_\nu(z)$ are the Bessel functions.

Then for $\mathrm{Re}\, s > 0$ Fubini's theorem gives

$$\int_0^\infty dt \frac{\chi(1)\mathrm{vol}(\Gamma\backslash G)}{4\pi}\int_{\mathbf{R}} dr e^{-(r^2+b_++\rho_0^2)t}|C(r)|^{-2}I_{s-\frac{1}{2}}(b_-t)t^{s-\frac{1}{2}}$$

$$= (2b_-)^{\frac{1}{2}-s}\frac{\chi(1)\mathrm{vol}(\Gamma\backslash G)\Gamma(s)}{4\pi^{3/2}}\int_{\mathbf{R}} dr |C(r)|^{-2}\prod_p(r^2+B_p)^{-s}, \qquad (3.9)$$

for $B_p = \rho_0^2 + b_p$. In order to analyze the last integral in Eq. (3.9) (for the possibility of a meromorphic continuation) it is useful to rewrite the function $|C(r)|^{-2}$ (see Eq. (2.5)), using the identity $\tanh(ar) \equiv 1 - 2(1+e^{2ar})^{-1}$. Then one can calculate a suitable integral in terms of the hypergeometric function $F(\alpha, \beta; \gamma; z)$, namely

$$\int_0^\infty dr r^{2j+1}\prod_p(r^2+B_p)^{-s} = \frac{\sqrt{\pi}\Gamma(2s-j-1)j!}{2^{2s}\Gamma(s)\Gamma(s+\frac{1}{2})}B_1^{-s}B_2^{j+1-s}\left(\frac{2B_1}{B_1+B_2}\right)^{j+1}$$

$$\times F\left(\frac{j+1}{2},\frac{j+2}{2};s+\frac{1}{2};\left(\frac{B_1-B_2}{B_1+B_2}\right)^2\right), \qquad (3.10)$$

which is a holomorphic function on $\mathrm{Re}\, s > (j+1)/2$ and admits a meromorphic continuation to $\mathbf{C}$ with only simple poles at points $s = (j+1-n)/2$, $n \in \mathcal{N}$.

For $Res > \frac{d}{4}$ the explicit meromorphic continuation holds [21]:

$$\zeta(s|\bigotimes_p \mathcal{L}_p) = A\sum_{j=0}^{\frac{d}{2}-1} a_{2j}\left(\mathcal{F}_j(s) - E_j(s)\right) + \mathcal{I}(s), \qquad (3.11)$$

where

$$E_j(s) \stackrel{def}{=} 4\int_0^\infty \frac{dr r^{2j+1}}{1+e^{2a(G)r}}\prod_p(r^2+B_p)^{-s}, \qquad (3.12)$$

which is an entire function of s and

$$A \stackrel{def}{=} \frac{1}{4}\chi(1)\mathrm{vol}(\Gamma\backslash G)C_G, \qquad (3.13)$$

$$\mathcal{F}_j(s) \stackrel{def}{=} (B_1B_2)^{-s}\frac{j!\left(\frac{2B_1B_2}{B_1+B_2}\right)^{j+1}F\left(\frac{j+1}{2},\frac{j+2}{2};s+\frac{1}{2};\left(\frac{B_1-B_2}{B_1+B_2}\right)^2\right)}{(2s-1)(2s-2)...(2s-(j+1))}, \qquad (3.14)$$

$$\mathcal{I}(s) \stackrel{def}{=} (2b_-)^{\frac{1}{2}-s}\frac{\sqrt{\pi}}{\Gamma(s)}\int_0^\infty dt\theta_\Gamma(t,b_+)I_{s-\frac{1}{2}}(b_-t)t^{s-\frac{1}{2}}. \qquad (3.15)$$

The goal now is to compute the zeta function and its derivative at $s = 0$. Thus we have

$$\mathcal{F}_j(0) = \frac{(-1)^{j+1}}{j+1}\left(\frac{2B_1}{B_1+B_2}\right)^{j+1} F\left(\frac{j+1}{2}, \frac{j+2}{2}; \frac{1}{2}; \left(\frac{B_1-B_2}{B_1+B_2}\right)^2\right)$$

$$= \frac{(-1)^{j+1}}{2(j+1)}\sum_l^2 B_l^{j+1}, \tag{3.16}$$

$$E_j(0) = 4\int_0^\infty \frac{dr\, r^{2j+1}}{1+e^{2a(G)r}} = \frac{(-1)^j}{j+1}(1-2^{-2j-1})\left[\frac{\pi}{a(G)}\right]^{2j+2}\mathcal{B}_{2j+2}, \tag{3.17}$$

$$\mathcal{I}(0) = 0, \tag{3.18}$$

where $\mathcal{B}_{2n}$ are the Bernoulli numbers.

Proposition 1 *A preliminary form of the zeta function* $\zeta(s|\bigotimes_p \mathcal{L}_p)$ *at* $s = 0$ *is*

$$\zeta(0|\bigotimes_p \mathcal{L}_p) = A\sum_{j=0}^{\frac{d}{2}-1}\frac{(-1)^{j+1}}{2(j+1)}a_{2j}\left\{\sum_l^2 B_l^{j+1} + (2-2^{-2j})\left[\frac{\pi}{a(G)}\right]^{2j+2}\mathcal{B}_{2j+2}\right\}. \tag{3.19}$$

Proposition 2 *The derivative of zeta function at* $s = 0$ *has the form:*

$$\zeta'(0|\bigotimes_p \mathcal{L}_p) = A\sum_{j=0}^{\frac{d}{2}-1}a_{2j}\sum_l^4 \mathcal{E}_l, \tag{3.20}$$

where

$$\mathcal{E}_1 = j!(B_1^{j+1}+B_2^{j+1})\sum_{k=0}^j \frac{(-1)^{k+1}}{k!(j-k)!(j+1-k)!}, \tag{3.21}$$

$$\mathcal{E}_2 = B_2^{j+1}\left(\frac{B_1-B_2}{2B_1}\right)\frac{(-1)^j}{(j+1)!}\sum_{k=1}^\infty \frac{(j+k+1)!}{(k+1)!}\sigma_k\left(\frac{B_1-B_2}{B_1}\right)^k, \tag{3.22}$$

$$\mathcal{E}_3 = \log(B_1 B_2)\frac{(-1)^j}{2(j+1)}(B_1^{j+1}+B_2^{j+1}) - 4\int_0^\infty \frac{dr\, r^{2j+1}\log\left(\frac{r^2+B_1}{r^2+B_2}\right)}{1+e^{2a(G)r}}, \tag{3.23}$$

$$\mathcal{E}_4 \equiv \mathcal{I}'(s=0) = T_\Gamma(0, b_1, \chi_1) + T_\Gamma(0, b_2, \chi_2), \tag{3.24}$$

and

$$T_\Gamma(0, b_p, \chi_p) \stackrel{def}{=} \int_0^\infty dt\,\theta_\Gamma(t, b_p)t^{-1}, \qquad \sigma_k \stackrel{def}{=} \sum_{k=1}^n \frac{1}{k}. \tag{3.25}$$

After a standard functional integration the contribution to the Euclidean one-loop affective action can be written as follows

$$W^{(1)} = \frac{1}{2}\mathrm{logdet}(\bigotimes_p \mathcal{L}_p/\mu^2) = -\frac{1}{2}[\zeta'(0|\bigotimes_p \mathcal{L}_p) + \log\mu^2\zeta(0|\bigotimes_p \mathcal{L}_p)], \tag{3.26}$$

where μ^2 is a normalization parameter. As a result we have

$$W^{(1)} = -\frac{1}{2}A\sum_{j=0}^{\frac{d}{2}-1}a_{2j}\left[\sum_l^4 \mathcal{E}_l + \log\mu^2(\mathcal{F}_j(0) - E_j(0))\right], \tag{3.27}$$

where $\mathcal{F}_j(0), E_j(0)$ and $\mathcal{E}_l$ are given by the formulae (3.16), (3.17) and (3.21) - (3.24) respectively.

3.2 The Explicit Form of Anomaly

In this section we start with a conformal deformation of a (pseudo-) Riemannian metric and the conformal anomaly of the energy stress tensor. It is well known that (pseudo-) Riemannian metrics $g_{\mu\nu}(x)$ and $\tilde{g}_{\mu\nu}(x)$ on a manifold M are (pointwise) conformal if $\tilde{g}_{\mu\nu}(x) = \exp(2f)g_{\mu\nu}(x)$, $f \in C^\infty(\mathbf{R})$. For constant conformal deformations the variation of the connected vacuum functional (effective action) can be expressed in terms of the generalized zeta function related to an elliptic self-adjoint operator $\mathcal{O}$ [18]

$$\delta W = -\zeta(0|\mathcal{O})\log\mu^2 = \int_M dx <T_{\mu\nu}(x)> \delta g^{\mu\nu}(x), \qquad (3.28)$$

where $<T_{\mu\nu}(x)>$ means that all connected vacuum graphs of the stress- energy tensor $T_{\mu\nu}(x)$ are to be included. Therefore the Eq. (3.28) leads to

$$<T^\mu_\mu(x)> = (\mathrm{Vol}M)^{-1}\zeta(0|\mathcal{O}). \qquad (3.29)$$

In the case of sphere S^d of unit radius we have for example $(\mathrm{Vol}S^d) = 2\pi^{(d+1)/2}/\Gamma((d+1)/2)$, while the Eqs. (3.2) and (3.3) give $VC_G\pi = (\mathrm{Vol}M)[(4\pi)^{d/2}\Gamma(d/2)]^{-1}$ (see for detail Ref. [27]). As a result we have $(\mathrm{Vol}M) = A(4\pi)^{d/2}\Gamma(d/2)$.

The formulae (3.11), (3.16), (3.17) and (3.18) give an explicit result for the conformal anomaly, namely

$$<T^\mu_\mu(x)>_{(\mathcal{O}=\otimes\mathcal{L}_p)} = \frac{1}{(4\pi)^{d/2}\Gamma(d/2)}\sum_{j=0}^{\frac{d}{2}-1}\frac{(-1)^{j+1}}{2(j+1)}a_{2j}$$

$$\times\left\{\sum_l^2 B_l^{j+1} + (2 - 2^{-2j})\left[\frac{\pi}{a(G)}\right]^{2j+2}\mathcal{B}_{2j+2}\right\}, \qquad (3.30)$$

where d is even.

For $B_1 = B_2 = B$ the anomaly (3.30) is associated with Laplace type operator $\mathcal{L} = L + b$ and has the form

$$<T^\mu_\mu(x)> = \frac{1}{(4\pi)^{d/2}\Gamma(d/2)}\sum_{j=0}^{\frac{d}{2}-1}\frac{(-1)^{j+1}}{2(j+1)}a_{2j}\left\{B^{j+1} + (2 - 2^{-2j})\left[\frac{\pi}{a(G)}\right]^{2j+2}B_{2j+2}\right\}. \qquad (3.31)$$

Note that for minimally coupled scalar field of mass m, $B = \rho_0^2 + m^2$.

The simplest case is, for example $G = SO_1(2,1) \simeq SL(2,\mathbf{R})$; besides $X = H^2$ is a two-dimensional real hyperbolic space. Then we have $\rho_0^2 = \frac{1}{4}, a_{20} = 1, C_G = 1, a(G) = \pi, |C(r)|^{-2} = \pi r \tanh(\pi r)$, and finally

$$<T^\nu_\mu(x \in H^2)> = -\frac{1}{4\pi}(b + \frac{1}{3}). \qquad (3.32)$$

For real d-dimensional hyperbolic space $C_G = [2^{d-2}\Gamma(d/2)]^{-2}$, while the scalar curvature is $R(x) = -d(d-1)$. For the conformally invariant scalar field we have $B = \rho_o^2 + R(x)(d-2)/[4(d-1)]$. As a consequence, for all constant curvature spaces $B_1 = B_2$; for hyperbolic spaces $B = \frac{1}{4}$ and

$$<T^\mu_\mu(x \in H^d)> = \frac{1}{(4\pi)^{d/2}\Gamma(d/2)}\sum_{j=0}^{\frac{d}{2}-1}\frac{(-1)^{j+1}}{j+1}a_{2j}\left\{2^{-2j-2} + (1 - 2^{-2j-1})B_{2j+2}\right\}. \qquad (3.33)$$

Thus in conformally invariant scalar theory the anomaly of the stress tensor coincides with one associated with operator product. This statement holds not only for hyperbolic spaces considered above but for all constant curvature manifolds as well.

4 Conclusions

In this paper the one-loop contribution to the effective action (3.27) and the conformal anomaly of the stress-energy momentum tensor (3.30) related to the operator product have been evaluated explicitly. In addition we have considered the product $\bigotimes_p \mathcal{L}_p$ of Laplace type operators $\mathcal{L}_p$ acting in irreducible rank 1 symmetric spaces. As an example the conformal anomaly has been computed for real d-dimensional hyperbolic spaces.

We have shown that for the class of constant curvature manifolds the conformal anomalies associated with the Laplace type operator $\mathcal{L}$ and the product $\bigotimes_p \mathcal{L}_p$ coincide. Our formulae can be generalized to the case of transverse and traceless tensor fields and spinors in real hyperbolic spaces (see, for example, [32, 33, 34]). Indeed in the case of the spin-1 field (vector field theory) for instance we should draw attention to the fact that the Hodge-de Rham operator $-(d\delta + \delta d)$ acting on co-exact one-forms corresponds to the mass operator $(L + d - 1)g_{\mu\nu}(x)$. The eigenvalues of this operator are $\lambda_l + (\rho_0 - 1)^2$, and for the Proca field of mass m we have $B = (\rho_0 - 1)^2 + m^2$. Finally we have also computed the anomaly in a simple situation, namely for the conformally invariant scalar fields. Our result (3.33) coincides with the quantum correction reported in Ref. [25] for compact hyperbolic spaces. Recently the conformal anomaly of dilaton coupled matter in four dimensions has been calculated in Refs. [35, 36]. It would be of great interest to generalize our results to the dilaton dependent trace anomaly.

An extension of the above evaluation of the effective action and the conformal anomaly for higher spin fields seems to us certainly feasible. The analysis of multiplicative properties of Laplace type operators and related zeta functions will be interesting in view of future applications to concrete problems in quantum field theory and for mathematical applications as well.

Acknowledgements

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